SP-1017/008/01

The Status of Developments in the Theory

of Stochastic Duels - II

C. J. Ancker, Jr.

28 September 1966

SDC

$\mathbb{SP}$ *a professional paper*

The Status of Developments in the Theory

of Stochastic Duels - II

by

C. J. Ancker, Jr.

28 September 1966

SYSTEM

DEVELOPMENT

CORPORATION

2500 COLORADO AVE.

SANTA MONICA

CALIFORNIA

The Status of Developments in the Theory of

Stochastic Duels - II

by

C. J. Ancker, Jr.
System Development Corporation
Santa Monica, California

ABSTRACT

This paper contains a brief outline of the history and the principal

features of the Theory of Stochastic Duels, as well as the major analytical

results to date. Areas for future research and possible fields of application

are also considered. The connection with other models of combat, such as

Lanchester's equations and games of strategy, is discussed. An annotated

bibliography is included.

I. INTRODUCTION

There have been many attempts in the past to come to grips with the problem of an analytical formulation of combat operations. In modern times, apparently the first serious analysis was made by Frederick William Lanchester [1][1] in 1916. Essentially, Lanchester's equations deal with the losses on opposing sides where large numbers are involved and where various assumptions about the loss rates are made. This, then, is a macroscopic view. Another major combat model uses game theoretic techniques and is exemplified by Dresher's book [2]. The essential feature of this model is the choice of an optimal course of action. That is, the major concern in game theory models is decision-making prior to the beginning of the fire fight. There are other mathematical analyses of combat in the literature (many of which are very special and apply only to particular situations), but these are perhaps the two best-known and best-developed. In any event, they illustrate the scope of such analyses and serve quite well as bench marks for orienting ourselves in the following discussion.

Von Clausewitz [3] remarked long ago that "war is nothing but the duel on a larger scale." At any rate, it is in this spirit that we shall proceed. As we shall see presently, the Theory of Stochastic Duels is concerned with the microscopic features of combat such as _individual_ kill probabilities, time between rounds fired, ammunition limitations, cover, concealment, surprise, mobility and so forth. This is in sharp distinction to Lanchester's theory, which aggregates all these effects. However, these two viewpoints are not

---

[1]Numbers in brackets [] refer to entries in the List of References at the end.

antithetical and, in fact, as has been shown in {6}[2], one may proceed from the microscopic to the macroscopic. The point is that microscopic parameters and variables are by far the most easily measured and verified, and consequently a theoretical relationship with the macroscopic helps to interpret and lend credence to the macroscopic results. More will be said on this later. Also, we are concerned only with the progress of the fire fight after it begins, that is, after the prior tactical and strategic decisions have been made. This contrasts with games of strategy, which are primarily concerned with choices of courses of action. Again, however, we shall see at least an embryonic relationship between these two viewpoints {5}.

The author first conceived of developing a theoretical structure for simple combat situations resembling one-versus-one duels in 1955 while employed by the Operations Research Office of Johns Hopkins University (now the Research Analysis Corporation). This interest was doubly motivated: First, he was struggling with a complicated problem involving combat between small infantry units and small armored units; the extreme complexity of this situation led him to seek an elementary model which might conceivably be attacked analytically with some success and then used as a building block to construct models of more complicated situations. As a second motivating force, it came to the author's attention at about that time that the Operations Research Office was comparing and evaluating certain weapon systems by simulated one-versus-one duels run on a hand-computation Monte Carlo basis using field data. The great gain in efficiency if an analytical solution were possible was obvious.

---

[2]Numbers in braces {} refer to the Annotated Bibliography on Stochastic Duels at the end.

Consequently, the author and a colleague, G. Trevor Williams, started an investiga-
tion which has been continued on an intermittent basis ever since. The first

general results were presented at the Fourth Annual Meeting of the Operations

Research Society of America in Washington, D. C., on 11 May 1956. In 1963

another colleague, A. V. Gafarian, became interested and has since made important

contributions. In the last year or so, the Indian analysts Jaiswal, Bhashyam,

and Singh {14}, {15} have made interesting and novel studies which are to be

published soon. Groves {11} and Schoderbek {1}, {2} have also contributed to

the basic theory. An annotated bibliography of all published papers and reports

on the theory known to the author is given at the end of this paper.

II. THE MODEL AND SUMMARY OF PRINCIPAL RESULTS

A. The Fundamental Duel and the Limited Fundamental Duel.

Figure 1 is a schematic representation of the fundamental model of the

duel. It consists of two contestants, A and B, who fire at each other with
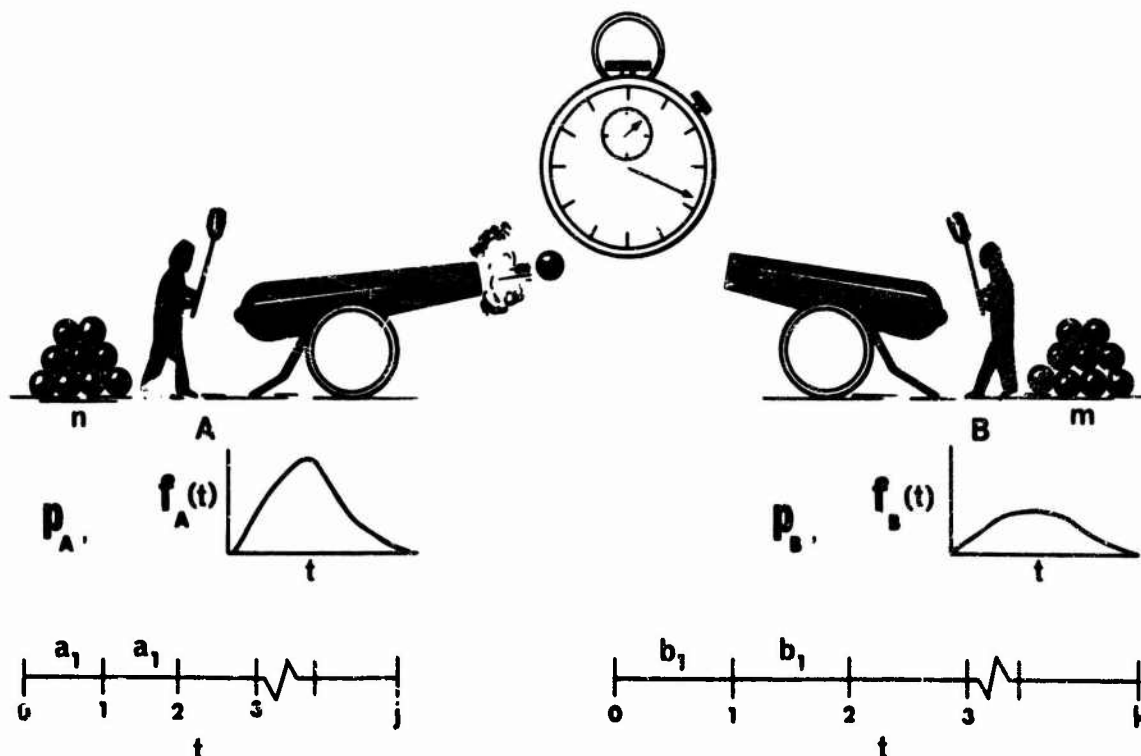

Figure 1. The Fundamental Model of the Duel and Some Modifications

fixed kill probabilities p_A and p_B respectively until one or both are hit. The
duelists begin simultaneously with unloaded weapons and fire at intervals of
time that are either random and characterized by continuous probability density
functions, $f_A(t)$ and $f_B(t)$, or are constants, a_1 and b_1, for A and B respectively.
Both contestants have unlimited ammunition supplies and unlimited time to score
a kill.

The general solution to the continuous—firing-time duel and to much of
what follows depends on two key concepts. The first is that this duel is
mathematically equivalent to a different situation in which all features are
the same except that the two contestants fire _independently_ at _separate_ targets
until each scores a hit. The one who hits his target in the least elapsed
time is declared the winner. The second basic concept concerns the convolution
property of characteristic functions. Consider the case where A fires at a passive
target. His probability of hitting the target in the time interval $[t,t+dt]$ on
the n$\underline{th}$ round fired from the beginning of the action is, clearly, $p_A q_A^{n-1} f_A(t)^{n*} dt$
where $f_A(t)^{n*}$ means n iterated convolutions of $f_A(t)$ $\left(e.g.\ f_A(t)^{2*} = f_A(t)*f_A(t)\right)$.
This expression reflects the fact that he must miss on the first n−1 rounds,
hit on the n$\underline{th}$ round, and that his time-to-hit will be the sum of n random
choices from his firing-time density function. Thus the total probability of
a hit in the interval $[t,t+dt]$, designated $h_A(t)dt$, is

$$h_A(t)dt = \sum_{n=1}^{\infty} p_A q_A^{n-1} f_A(t)^{n*} dt \ . \tag{1}$$

From the convolution property of characteristic functions, equation (1) may be
immediately transformed to

$$\Phi_A(u) = \sum_{n=1}^{\infty} p_A q_A^{n-1} \phi_A^n(u) = \frac{p_A \phi_A(u)}{1 - q_A \phi_A(u)} , \tag{2}$$

where $\Phi_A(u)$ and $\phi_A(u)$ are the characteristic functions of $h_A(t)$ and $f_A(t)$ respectively and $q_A = 1 - p_A$. This very considerable simplification is the second key concept.

The general solution for the fundamental duel consists of the probability of a given side winning, e.g., $P(A)$ is the probability A wins. The general solution for the duel with continuous random firing-times has been shown to be (see {3})

$$P(A) = \frac{1}{2} + \frac{1}{2\pi i} (P) \int_{-\infty}^{+\infty} \Phi_A(-u) \Phi_B(u) \frac{du}{u} = \frac{1}{2\pi i} \int_L \Phi_A(-u) \Phi_B(u) \frac{du}{u} \tag{3}$$

where $(P) \int_{-\infty}^{+\infty}$ means the Cauchy principal value of the integral and $\int_L$ means the limit as $R \to \infty$ of the integral around the contour in the complex u-plane of Figure 2(b) where $\rho > 0$ but less than t distance to the nearest singularity. Of
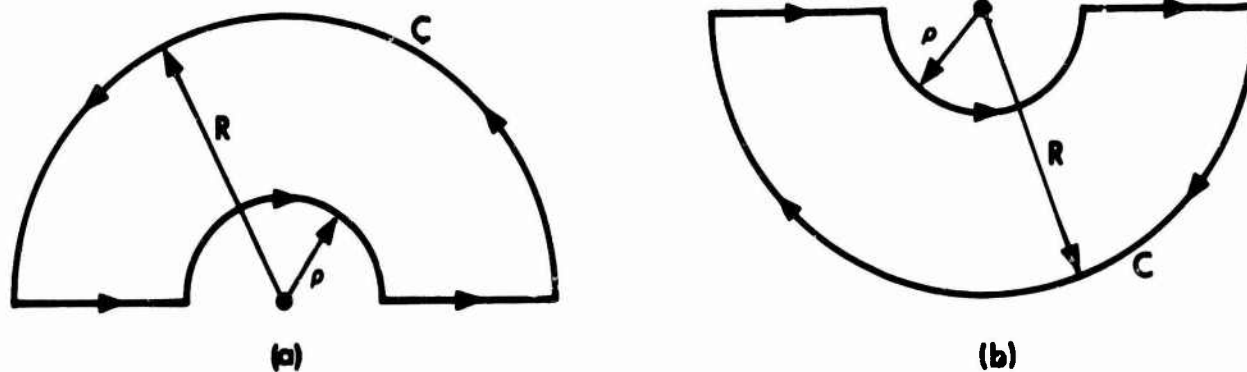

Figure 2. Paths of Integration for $\int_U$ and $\int_L$

course, the integral on C must be zero in the limit. An example of the evaluation of (3) for the specific case where the firing-time density functions, $f_A(t)$ and $f_B(t)$, are $(\gamma,2)$ variates is graphically illustrated as a set of contour maps in Figure 3, where r_A and r_B are rates of fire and are defined as the reciprocals of the mean times-to-fire.

The general solution to the fundamental duel with fixed firing-times a_1 and b_1 is (see {5})

$$P(A) = \frac{p_A}{1-q_A^b q_B^a} \sum_{j=0}^{b-1} q_A^j \; q_B^{[(j+1)\frac{a}{b}]} , \tag{4}$$

where the notation [x] means the largest integer less than or equal to x. This solution assumes that the ratio a_1/b_1 is rational and that if a_1 and b_1 contain a common factor it is reduced to a/b where a and b are relatively prime. If a_1 and b_1 are relatively prime, then $a_1=a$, $b_1=b$. Equation (4) is presented graphically in Figure 4.

One reason that Figures 3 and 4 appear to be so different is that in the fixed firing-time case a double kill (draw) is possible (and thus the possibility of a win by A is reduced) whereas this cannot happen in the continuous firing-time case.

Similar expressions have been derived for the fundamental duel with either a limitation on initial ammunition supplies (see {4}) or a limitation on the time-duration (see {8}) of the duel. To illustrate the nature of these expressions, one form of the general solution for each kind of limitation with continuous firing-times only will be given. For the ammunition limitation
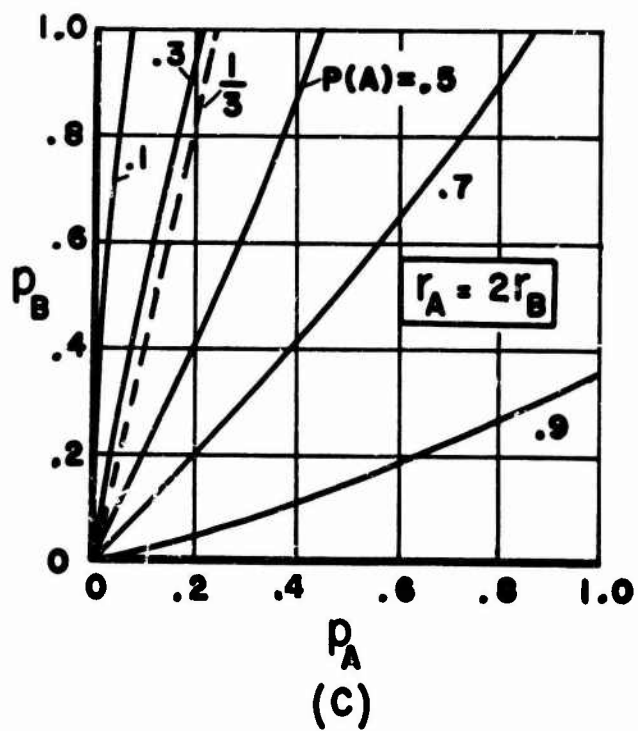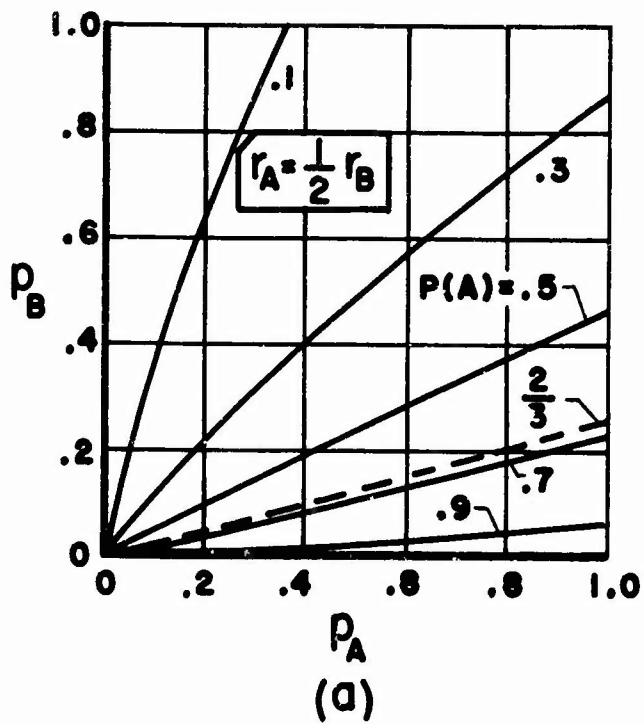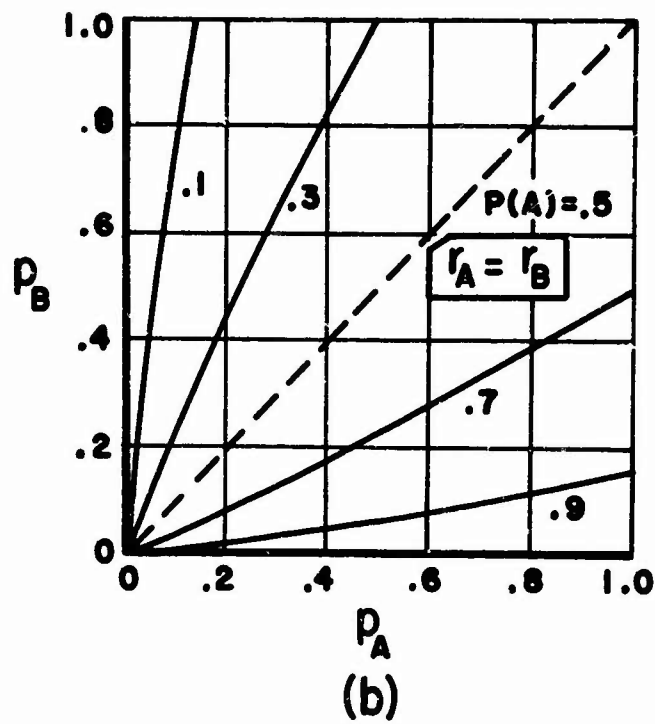
Figure 3. The Fundamental Duel with (γ,2) Firing-Times
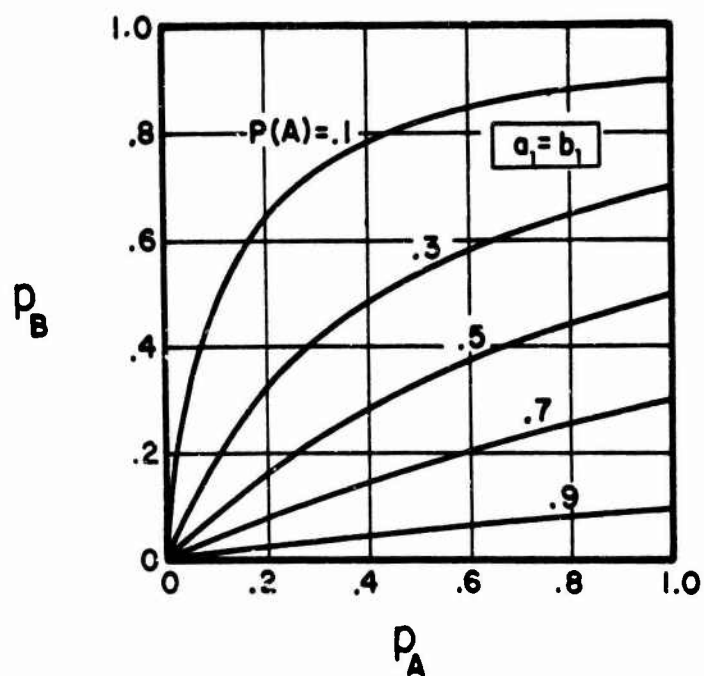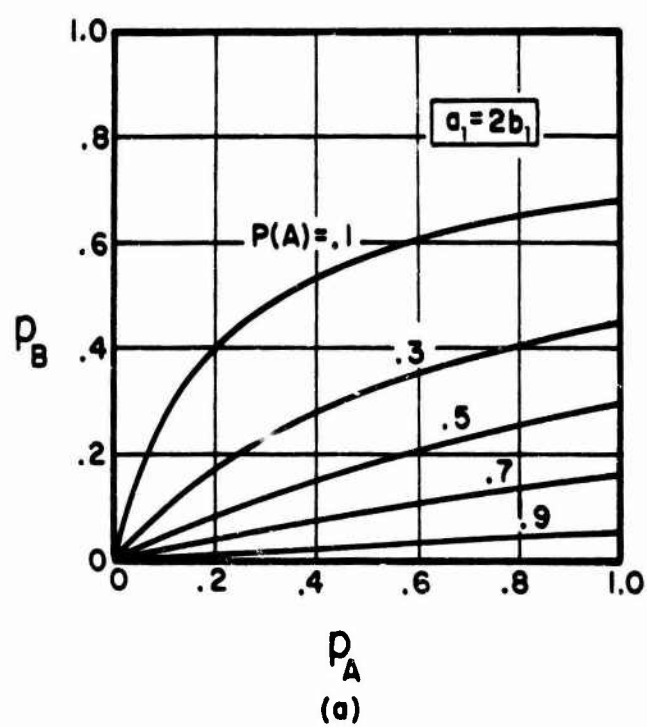
Figure 4.　The Fundamental Duel with Discrete Firing-Times

situation, let A start with n rounds and B with m rounds only, and let the probability distributions of n and m be

$$\left.\begin{array}{l} P\{n=k\} = \alpha_k \ , \ k=0,1,2,\ldots \\[2mm] P\{m=j\} = \beta_j \ , \ j=0,1,2,\ldots \\[2mm] \sum_{k=0}^{\infty} \alpha_k = \sum_{j=0}^{\infty} \beta_j = 1 \end{array}\right\} \qquad , \qquad (5)$$
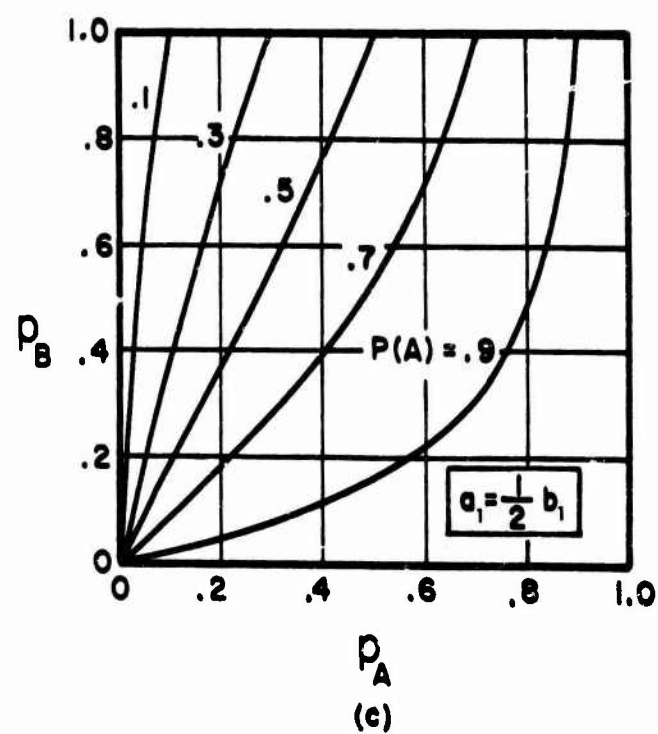
then from (4)

$$P(A) = \sum_{j=0}^{\infty} \beta_j q_B^{\ j}\left[1 - \sum_{k=0}^{\infty} \alpha_k q_A^{\ k}\right] + \frac{1}{2\pi i}\int_L \Phi_{A1}(-u)\Phi_{B1}(u)\frac{du}{u} , \qquad (6)$$

where

$$\Phi_{A1}(u) = \frac{P_A \phi_A(u)}{1-q_A \phi_A(u)}\left\{1 - \sum_{j=0}^{\infty} \beta_j [q_A \phi_A(u)]^j\right\} , \qquad (7)$$

and similarly for Φ_{B1}. Figure 5 illustrates the effect of ammunition limitation on the outcome of the duel. In the example shown, both sides have exponential firing-times with rates of fire r_A and r_B respectively. $P(A)_U$ is the solution for an entirely unlimited duel and $P(A)$ is the outcome for the duel where A has a <u>fixed</u> limit of n rounds and B is unlimited. The variable x in the figure is $\frac{q_A r_A}{p_B r_B + r_A}$. The figure clearly shows how an ammunition limitation may greatly affect the outcome of the duel.
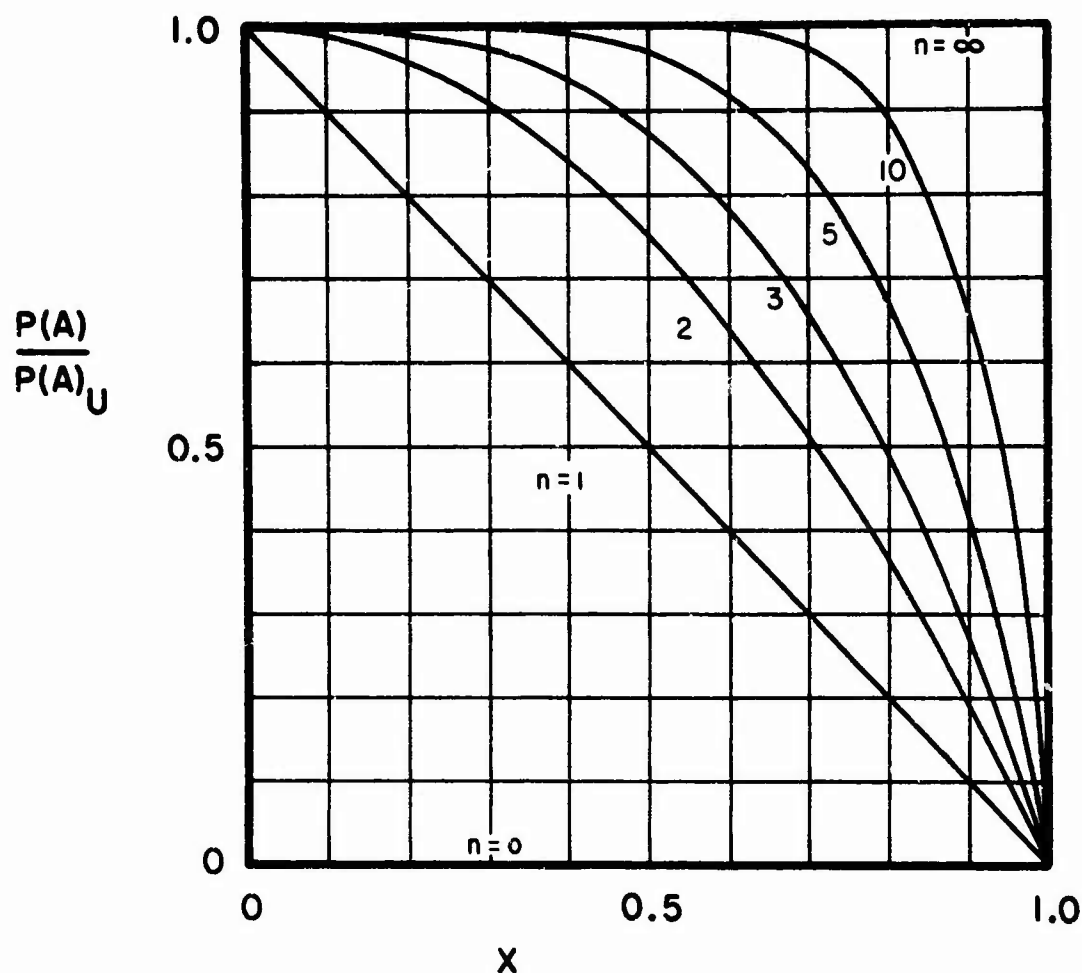
Figure 5. The Effect on A of Ammunition Limitation on A

If the time-duration of the duel is limited and is a random variable whose probability density function is $g(t)$, then the general solution (see {8}) is

$$P(A) = \frac{1}{4\pi^2} \int_U \frac{\theta(-u)}{u} \left(\int_L \frac{\Phi_A(u-w)\Phi_B(w)\,dw}{w} \right) du, \tag{8}$$

where $\theta(u)$ is the characteristic function of $g(t)$ and the path of integration for $\int_U$ is shown in Figure 2(a). The effect of time-limitation is shown in Figure 6 where again the limited and unlimited duels are compared. All firing-times are
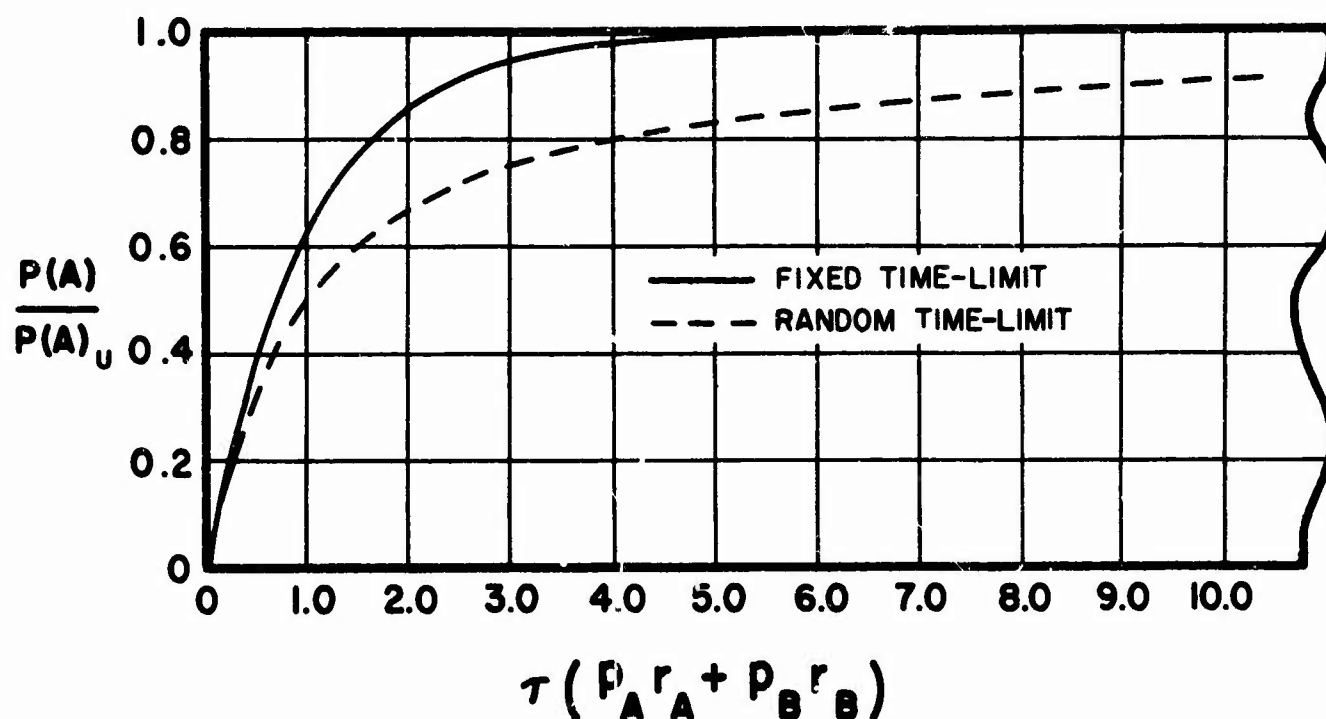
Figure 6. The Effect of Time – Limitation on the Outcome
of a Random Firing – Time Duel

exponential, with rates r_A and r_B, and the random time-limit is exponential with
mean τ. The fixed time-limit curve is for the duel where the random limit has
been replaced a fixed limit τ. Again, the effects of limitation are obviously
considerable if the limitation is moderate or severe.

The distribution of the time-to-completion has been derived {10} for
unlimited and time-limited duels, and the distribution of the total number of
rounds fired has been determined {7} for unlimited and ammunition-limited duels.
Also, as a by-product, at various places along the way, all of the foregoing has
been worked out for the marksman firing at a passive target.

In {6}, two approximations to the unlimited fundamental duel have been derived for arbitrary continuous firing-time distributions so that one may take field data on this variable and apply it directly rather than attempting to select, by curve-fitting, a theoretical distribution for use in the exact solution.

The extent of the completed work on the fundamental duel, the limited fundamental duel, and the marksman problem is indicated in Figure 7. Specific examples are given for all the general results shown.

B. Modifications and Extensions to the Fundamental Duel

In addition to the results outlined above, certain other modifications and extensions to the fundamental duel have been made. One of these is the addition of the effect of concealment or surprise. This extension is accomplished by assuming that one side or the other is allowed to fire before the opponent is alerted and returns the fire. In one case we have assumed that each contestant gets this advantage as a single "free" shot one-half the time and we have called this "tactical equity" (see {3}). Far more generally, however, one may assume that A or B gets a time-advantage during which he may fire with impunity. If this time is taken to be a continuous random variable with characteristic function $\psi(u)$ then the solution is (see {3})

$$P(A) = \frac{1}{2} + \frac{1}{2\pi i}(P)\int_{-\infty}^{+\infty} \Phi_A(-u)\Phi_B(u)\psi(u)\frac{du}{u} \approx \frac{1}{2\pi i}\int_L \Phi_A(-u)\Phi_B(u)\psi(u)\frac{du}{u} \ . \ (9)$$

The effect of surprise is illustrated in Figure 8, where the solution for the fundamental duel with exponential firing-times (rates of fire, r_A and r_B) is plotted in Figure 8(a) and the corresponding duel with normally distributed (zero mean, variance σ) surprise-time is shown in 8(b). This example distributes

	Unlimited					Ammunition-Limited (2)					Time-Limited (3)				
	P(A) etc. (1)	Time-to-completion Distr.	Moments	Rounds Fired Distr.	Moments (4)	P(A) etc. (1)	Time-to-completion Distr.	Moments	Rounds Fired Distr.	Moments (4)	P(A) etc. (1)	Time-to-completion Distr.	Moments	Rounds Fired Distr.	Moments
MARKSMAN Continuous Firing-Time	///	$5^{(5)}$ $6A^{(6)}$	10	7	7	7			7	7	10	10	10		
Fixed Firing-Time	10	10	10	7	7	7			7	7	10	10	10		
FUNDAMENTAL DUEL Continuous Firing-Time	3 $6A^{(6)}$	10	10	7	7	$4^{(9)}$			7	7	8	10	10		
Fixed Firing-Time	5	10	$10^{(7)}$	7	7	7			7	$7^{(8)}$	8	10	$10^{(7)}$		

(1) includes P(B) and P(AB) (draws) where appropriate

(2) includes both the random (discrete) and fixed limitation cases

(3) includes both the random (continuous) and fixed limitation cases

(4) includes the right tails

(5) NUMBERS IN THE TABLE REFER TO SOURCE OF THE SOLUTION AS GIVEN IN THE ANNOTATED BIBLIOGRAPHY

(6) A means approximation

(7) computable but no useful simplification found

(8) no useful simplification for discrete distribution of ammunition limitation

(9) includes case of duel terminating when unkilled contestant runs out of ammunition

Figure 7. General Solutions Available to the Fundamental Duel, The Limited Fundamental Duel, and the Marksman Problem
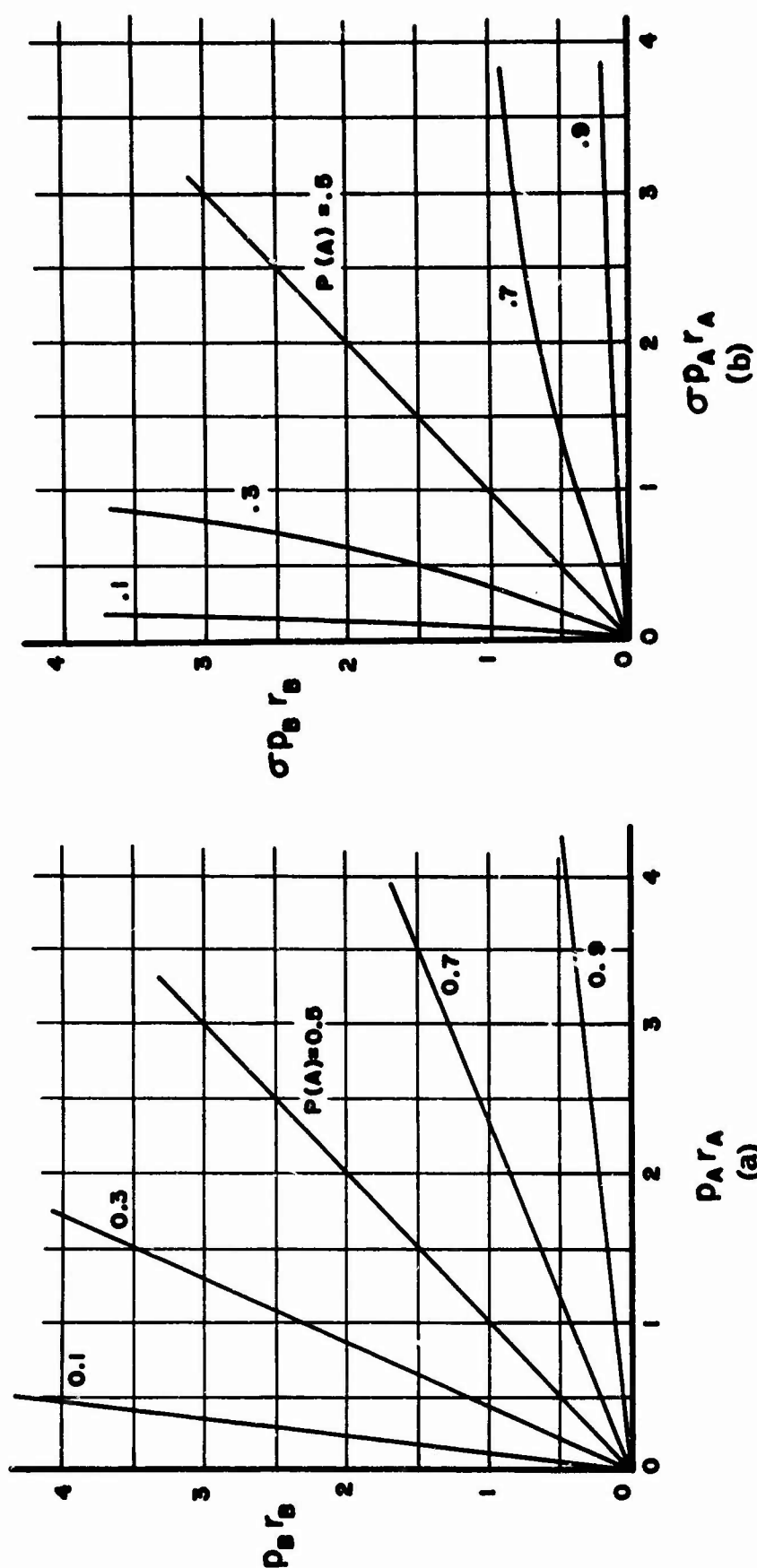
Figure 8. Comparison of the Fundamental Exponential
Duel without and with Surprise

the advantage equally to both sides; the equal distribution accounts for the fact that in the wedge above the 45° line A's chances are enhanced by surprise and in the lower wedge they are reduced. Some related results for the fixed firing-time case are given in {1} and {11}.

Various other modifications to the fundamental duel are summarized in Figure 9. The duel with variable kill probabilities investigates the effect on the outcome of having both contestants "zero" in on the target from round to round. The classical duel (where each contestant starts with a loaded weapon) shows how initial conditions may rather drastically affect the outcome.

The duel with displacements considers a model in which A and B fire simultaneously and either may score a "near miss," causing the opponent to displace and lose a firing turn. Some other early work on mobility, in which the amount of displacement and the opponent's radius of damage and dispersion are important, was done by Schoderbek {1}, {2}.

Another extension includes the effect of the time-of-flight of the projectiles. If time-of-flight is fixed (τ_A for A and τ_B for B) and if the contestants fire as rapidly as possible (i.e. without delay between rounds), then the solution is (see {13})

$$P(A) = \frac{1}{2\pi i} \ (P)\int_{-\infty}^{+\infty} \exp[-i\tau_A u]\phi_A(-u)\phi_B(u)du/u$$

$$= \frac{1}{2\pi i} \int_{L} \exp[-i\tau_A u]\phi_A(-u)\phi_B(u)du/u$$

(10)

or, for exponential firing-times,

$$P(A) = \left(\frac{p_A r_A}{p_A r_A + p_B r_B}\right) \exp[-p_B r_B \tau_A] \quad .$$

(11)

SIMPLE (ONE-VERSUS-ONE) DUELS

1. <u>Variable Kill Probabilities</u> –

    a. p_A and p_B are monotonic increasing functions of round number {9}.

    b. p_A and p_B are special functions of time, exponential firing-times only {2}.

    c.   Round dependent kill probabilities, exponential firing-times, limited and unlimited ammunition {14}.

2. <u>Classical Duel</u> – contestants start with loaded weapons {3}.

3. <u>Duel With Initial Surprise</u> –

    a.   Tactical equity–each side gets a "free" shot one-half the time {3}.

    b.   Random initial surprise {3}.

    c.   A has arbitrary time-advantage, fixed firing-times {1}, {11}.

4. <u>Displacement</u> –

    a.   A and B fire at equal fixed intervals with a "near-miss" causing a displacement (and loss of firing turn) {5}.

    b.   A's survival probability on next round fired when B's round-to-round dispersion and radius-of-kill are given and

        (1) A displaces after firing k^{th} round – fixed firing-times {1}

        (2) A displaces at time t_o –continuous firing-times {2}

5. <u>Time-of-Flight</u> –

    a.   With delay and no-delay between rounds, also linearly varying with time, {13}

    b.   No-delay, fixed ammunition supply, fixed replenishment–continuous firing-times {15}

NOTE: (Numbers in braces { } refer to the Annotated Bibliography)

Figure 9. Modifications and Extensions to the Fundamental Duel

$Exp[-p_B r_B \tau_A]$ is the effect of time-of-flight and can obviously be serious. Several other time-of-flight models are contained in {13}, including a procedure in which the firer waits until his round lands before preparing and firing his next round, and the general solution to the duel when times-of-flight vary linearly with time.

Recently, some new techniques have been introduced by certain Indian authors, notably N. K. Jaiswal, N. Bhashyam and N. Singh. In {14} the use of the differential-difference technique is introduced. It is applied to the exponential firing-time, fundamental duel with fixed ammunition supply and variable kill probabilities (round dependent). In {15} the use of the Keilson-Kooharian supplementary variable technique occurs for the first time and is applied to the continuous fundamental duel with random time-of-flight included and fixed ammunition replenishment at exponential intervals.

In still another interesting paper {11}, Groves has introduced the use of the theory of Markov processes to analyze the fundamental duel with fixed firing times and with one contestant beginning a fixed amount of time earlier than the second.

C. Duels with More Than Two Contestants

Some models with more than two contestants are summarized in Figure 10.

The duel with displacements (mentioned above) and the several multiple duels are a first attempt at considering situations in which the actions on the A side are not independent of the actions on the B side. Up to this point we have been able to uncouple the actions of the A and B sides (at least up to the point of a kill), but in these duels it is no longer possible to do this.

DUELS WITH MORE THAN TWO CONTESTANTS

1. <u>Several Multiple Duels</u> - All fire at fixed intervals, all p_A's equal. All p_B's equal.

   a. Triangular - two A's versus one B.   {5}

   b. First Square Duel $\begin{Bmatrix} A \searrow & \leftarrow B \\ A \rightarrow & \leftarrow B \end{Bmatrix}$ {5}

   c. Second Square Duel $\begin{Bmatrix} A \rightarrow & \leftarrow B \\ A \rightarrow & \leftarrow B \end{Bmatrix}$ {5}

   d. Cluster Duel

      (1) Many or each side - outcome of a single firing, {5}, {19}.

      (2) Same - round dependent kill probabilities, approximations,
           connections with Lanchester's Models {20}.

2. <u>Sequential Duels</u> - Fundamental Duel, exponential firing-times, A
   fights a sequence of B's until defeated {2}.

3. <u>Two Large Battles</u> - {6} - Many one each side, all p_A's equal, all p_B's equal.
   Exponential firing-times all the same on each side.

   a. All visible and attack each other at all times → Lanchester's square law.

   b. Only one pair at a time duel → Lanchester's linear law.

NOTE: (Numbers in braces { } refer to annotated Bibliography).

Figure 10. Extensions of the Fundamental Duel to Multiple
Contestant Situations

Con3equently, one must consider each event as it occurs, as well as all the

possible interactions and conditional events which may occur subsequently. For

this purpose we have assumed very simple situations (all firing simultaneously,

for example) and enumerated all possible outcomes. In comparing the first and

second square duels (two on each side) with both A's concentrating on one B

in the first and each A attacking separate B's in the second, we are considering

strategy for the first time. The analysis shows that A's strategy in the second

case is superior to his strategy in the first.

In the Cluster duel {5} arbitrarily large numbers are engaged on each side,

all are intervisible, and all fire together. We compute the outcome of a

single firing. Robertson {19} did some early work on this situation and

Helmbold {20} extended it to variable kill probabilities from round to round.

He also developed many useful approximations and showed how this model leads

to the Lanchester models.

In the two large battles, Williams has shown some interesting connections

with the Lanchester equations. In the first, where all contestants are targets

for the opposition at all times and all contestants on the A side have the same

fixed kill probability and the same exponential time-to-fire distributions,

and similarly for B, the first order approximation to the solution (for large

numbers on each side) is Lanchester's square law. In the second battle only

one member of each side is in combat at any given time (with the other

assumptions the same as in the first battle). This model is analogous to two

columns meeting head-on in a narrow defile. The corresponding approximation

gives Lanchester's linear law in this case.

It is most heartening to see the connection between the microscopic and macroscopic viewpoints mentioned above. This enhances the credibility of the Lanchester formulation and indeed gives an interpretation to the "relative efficiencies" in the latter model. This research is related to some work which has been done on stochastic formulations of Lanchester's Laws (see for example [4] and [5]).

III. <u>SOME SUGGESTIONS FOR FUTURE RESEARCH AND POSSIBLE APPLICATIONS OF THE THEORY</u>

It is clear from a glance at Figure 7 that much remains to be done on the fundamental duel alone. All the blanks in this chart are areas which should be investigated (e.g., the distribution of time-to-completion for the ammunition-limited duel). In addition, the solution to the duel in which <u>both</u> time and ammunition are limited would be desirable. Then again the whole question of approximations has only been touched on. This is important in using field data in the theoretical equations.

With regard to one-versus-one combat, numerous extensions and ramifications of the fundamental duel come to mind. The matter of variable kill probabilities needs further examination. At least two possibilities suggest themselves, namely, time-dependent kill probabilities (as in vehicles which are closing on each other) and randomly varying kill probabilities. The inclusion of surprise has been accomplished only for the unlimited duel with continuous firing-times and for special cases in the discrete firing-time duel. This should be pushed further to include general discrete firing-times and limitations on time and ammunition. Mobility has barely been touched on in extremely simplified cases.

The first step forward here might be to consider mobility in the fundamental
duel. Finally, one might attempt the duel between two contestants armed with
automatic weapons which fire in bursts of variable length (time between rounds
in a burst is fixed) with variable time between bursts.

In duels involving more than one contestant on each side, clearly even
more factors remain to be examined. It will be sufficient to suggest two
extensions, namely, the solution of the large battle with general continuous
random firing-times and the same battle with different but fixed firing-times.
Incidentally, it is in these larger contests that one may begin to ask questions
which have greater strategic significance.

We now turn our attention to possible applications of the theory. As
mentioned earlier, one of the initial motivations for undertaking these studies
was the fact that hand-played Monte Carlo duels have been used in comparing one
weapon system with another. Certainly, weapons system evaluation should continue
to be a major field of application. Wherever questions concerning the trade-offs
between such parameters as weapon accuracy, rates of fire, concealment, mobility
etc., are important, these models provide a convenient analytical tool. The
unlimited fundamental duel is a highly idealized model which may be satisfactory
in many situations; but if ammunition limitations are severe (as in armored
vehicles) or time is restricted (as in air combat with limited fuel supplies),
then the more sophisticated models are appropriate.

A second area of application is in evaluating tactics and strategy. An
example of this, given earlier, compared two square duels to answer the question--
should one concentrate one's fire? A great deal more may be done in this sort
of thing than has been accomplished so far.

Not only combat functions but combat-support and logistics functions may also be analyzed from these models. Clearly the distribution of the number of rounds fired during the fire fight is closely related to ammunition supply requirements and stockage build-ups necessary in rear areas. Similarly, the distribution of time-to-completion of the fire fight has supply and maintenance implications. For example, the mean and variance of this quantity are important in considering fuel stockpiling for aerial combat vehicles. Of course, these distributions have tactical implications as well, e.g., the expected duration of the fire fight might well be a consideration in choosing the time for beginning an attack.

Finally, one other military area of application comes to mind, namely system design. An example of this application is the following. In designing a new tank should (1) armor be increased? (cuts down enemy's kill probability), (2) engine power be increased? (increases mobility), (3) weapon accuracy be increased? (increases our kill probability), (4) rate of fire be increased? (increases our probability of winning), etc. None of these questions can be answered in a vacuum. All must be considered together with particular reference to the trade-offs in cost and effectiveness. Here is where our models should be of assistance.

Although we have turned our attention exclusively to military combat and combat-support matters, it is worthwhile to point out that our models are not necessarily interpreted in this way. They may also be used in appropriate civilian and noncombat military applications. For example, consider a situation in which one wishes to choose between two systems A and B. These systems break

down and require maintenance periodically. If the breakdown is severe enough,
re-air is uneconomical and the system must be scrapped and a new one purchased.
This may be interpreted as a duel in which the time between breakdowns is
analogous to the time between rounds fired, $f_A(t)$ and $f_B(t)$; the probability
of being unsuccessfully repaired on each breakdown corresponds to the kill
probabilities, p_A and p_B; (N. B. the roles of A and B must be reversed in
designating the $f(t)$'s and the p's, e.g. $f_B(t)$ is the distribution of the
intervals between A's breakdowns, and p_B is the probability that A is not
successfully repaired) and the outcome, $P(A)$ (A's probability of winning the
duel) is the probability that A breaks down permanently before B.

IV. CONCLUSION

We have examined the history and principal results to date for the Theory
of Stochastic Duels. In addition, the connections with other models of combat
have been examined and areas for future research have been suggested. Since
the theory is comparatively new and has only recently begun to appear in the
literature, few actual applications are presently known. However, several
obvious areas of applications are suggested here. An annotated bibliography
complete to date (to the best of the author's knowledge) follows the list of
references.

LIST OF REFERENCES

[1] F. W. Lanchester, <u>Aircraft in Warfare; the Dawn of the Fourth Arm</u>, Constable and Co., London, 1916.

[2] Melvin Dresher, <u>Games of Strategy-Theory and Applications</u>, Prentice Hall, Inc., Englewood Cliffs, N. J., 1961.

[3] Karl Von Clausewitz, <u>On War</u>, Barnes and Noble, New York, 1956 (Translated by J. J. Graham).

[4] Richard H. Brown, "Theory of Combat: The Probability of Winning," <u>Operations Research</u>, Vol. 11, No. 3, May-June 1963, pp. 418-425.

[5] David G. Smith, "The Probability Distribution of the Number of Survivors in a Two-Sided Combat Situation," <u>Operational Research Quarterly</u>, Vol. 16, No. 4, December 1965, pp. 429-437.

ANNOTATED BIBLIOGRAPHY ON STOCHASTIC DUELS

I. One-Versus-One Duels

Term "fundamental duel" implies two contestants, A and B, fixed kill probabilities, unlimited time, unlimited ammunition, and simultaneous start with unloaded weapons. Unless otherwise stated, all models have these assumptions with any exceptions or additions as noted. "Marksman" is similar except it implies one firer versus passive target.

{1} Joseph J. Schoderbek, "Some Weapon System Survival Probability Models--I.

Fixed Time Between Firings," <u>Operations Research</u>, Vol. 10, No. 2,

March-April 1962, pp. 155-167.

> Fundamental duel with fixed firing-times. B begins a fixed time after A. Special cases only. Mobility (evacuation), radius of kill, round-to-round dispersion concepts also introduced.

{2} Joseph J. Schoderbek, "Some Weapon System Survival Probability Models--II.

Random Time Between Firings," <u>Operations Research</u>, Vol. 10, No. 2,

March-April 1962, pp. 168-179.

> Fundamental duel, all exponential firing times, general solution. Special cases where p_A and p_B are functions of time. Mobility (evacuation) and round-to-round dispersion also. A fights a succession of individual battles until finally defeated.

{3} Trevor Williams and Clinton J. Ancker, Jr., "Stochastic Duels," <u>Operations</u>

<u>Research</u>, Vol. 11, No. 5, September-October 1963, pp. 803-817 (Also

available as System Development Corporation document - SP-1017/000/01,

20 March 1963, 22 pp.)

> All duels with continuous random firing-times. Fundamental duel. Classical duel (start with loaded weapons). Tactical equity duel (one "free" shot). Duel with continuous random initial surprise. Distribution of time-to-hit by marksman. General solutions, examples and solution maps.

{4} C. J. Ancker, Jr., "Stochastic Duels with Limited Ammunition Supply,"

Operations Research, Vol. 12, No. 1, January-February 1964, pp. 38-50.

(Also available as System Development Corporation document - SP-1017/001/00,

23 April 1963, 24 pp.)

> Continuous random firing-times. Fundamental duel with
> discrete (random) and fixed ammunition limitation.
> Special case of duel terminating when unkilled contestant
> runs out of ammunition. General solutions, examples and
> comparison curves.

{5} C. J. Ancker, Jr. and Trevor Williams, "Some Discrete Processes in the

Theory of Stochastic Duels," Operations Research, Vol. 13, No. 2,

March-April 1965, pp. 202-216. (Also available as System Development

Corporation document - SP-1017/002/00, 13 August 1963, 28 pp.)

> Fundamental duel with fixed firing-times. Duel with
> displacements (simultaneous firing, near miss causes
> loss of firing turn). Multiple duels with simultane-
> ous firings and common kill probabilities on each
> side; includes triangular duel, two square duels and
> cluster duel with many on each side (outcome of a
> single firing for latter). General results, solution
> maps and comparison curves.

{6} Trevor Williams, "Stochastic Duels - II," SP-1017/003/00, 13 September 1963,

System Development Corporation, Santa Monica, California. 61 pp.

> Connection between cumulants of time-to-fire and
> time-to-kill for marksman with continuous random
> firing-time. Two approximations to the fundamental
> duel with continuous, random firing-times. Approxi-
> mate solutions to two large battles with common
> exponential firing-times and kill probabilities on
> each side. First case (all contestants participate
> continually) leads to Lanchester's square law.
> Second case (only one member at a time on each side
> in contact) gives Lanchester's linear law. Examples.

{7} C. J. Ancker, Jr. and A. V. Gafarian, "The Distribution of Rounds Fired

In Stochastic Duels," <u>Naval Research Logistics Quarterly</u>, Vol. 11, No. 4,

December 1964, pp. 303-327. (Also available as System Development Corporation

document - SP-1017/004/00, 4 March 1964, 35 pp.)

> Continuous random and fixed firing-times with and without
> ammunition limitation. Distribution, first two moments and
> right tail of rounds fired. Includes solution to fundamental
> duel with fixed firing-times and ammunition limitation.
> Includes distributions, etc., and general solution to marksman
> problem with continuous random and fixed firing-times with
> and without ammunition limitation. General solutions,
> special cases and examples.

{8} C. J. Ancker, Jr., "Stochastic Duels of Limited Time-Duration," <u>Canadian</u>

<u>Operational Research Society (CORS) Journal</u>, Vol. 4, No. 2, July 1966,

pp. 69-81. (Also available as System Development Corporation document -

SP-1017/005/00, 30 March 1964, 24 pp.)

> Continuous random and fixed firing-times. Fundamental
> duel with continuous random or fixed time-limit.
> General solutions, examples and comparison curves.

{9} (a) Trevor Williams, "Stochastic Duels - III," SP-1017/006/00, 22 June 1964,

System Development Corporation, Santa Monica, California. 72 pp.

(b) Trevor Williams, "Stochastic Duels with Homing," SP-1017/106/00,

18 May 1965, System Development Corporation, Santa Monica, California. 34 pp.

> Fundamental duel with kill probabilities increasing
> monotonically from round to round by using Pascal
> distribution for conditional hit probabilities.
> Exponential firing-times. Maximum likelihood
> estimate of "homing" from field data. Solutions,
> special cases and solution maps.

{10} C. J. Ancker, Jr. and A. V. Gafarian, "The Distribution of the Time-Duration of Stochastic Duels," <u>Naval Research Logistics Quarterly</u>, Vol. 12, Nos. 3 & 4, September-December 1965, pp. 275-294. (Also available as System Development Corporation document - SP-1017/007/00, 10 August 1964 29 pp.)

> Continuous random and fixed firing-times with and without
> either continuous random or fixed time-limit. Distribution
> and all moments for time-to-completion. Includes distribu-
> tions and moments and general solutions to marksman problem.
> General solutions and examples.

{11} Arthur D. Groves, "The Mathematical Analysis of a Simple Duel,"
BRL report No. 1261, August 1964, Ballistic Research Laboratories,
Aberdeen Proving Ground, Maryland. 23 pp.

> Fundamental duel with fixed firing times. B begins a
> fixed time after A. General solution. Examples.
> Introduces Markov Theory.

{12} C. J. Ancker, Jr., "The Status of Developments in the Theory of Stochastic
Duels," <u>Proceedings 14th Military Operations Research Symposium,</u> Fall
1964, pp. 260-269.

> History. Principal features of models. Major results.
> Suggestions for future results and fields of applica-
> tion. Bibliography.

{13} C. J. Ancker, Jr., "Stochastic Duels with Time-of-Flight Included,"
<u>Opsearch</u>, Vol. 3, No. 2, 1966, pp. 1-22. (Also available as System
Development Corporation document -- SP-1017/009/00, 19 May 1966, 28 pp.)

Fundamental duel with time-of-flight included. No-delay
procedure with random firing-times (with either random or
fixed time-of-flight) and fixed firing-times (with
fixed time-of-flight). Delay procedure with random
firing-times or fixed firing-times (with random time-of-
flight). Mixed procedures. Linearly varying time-of-
flight, delay procedure, random firing-times. Marksman
with random or fixed firing-times and random or fixed
time-of-flight. General solutions and examples.

{14} N. Bhashyam and Naunihal Singh, "Stochastic Duels with Varying Single

Shot Kill Probabilities," Defence Science Laboratory, Delhi, to be

published.

Exponential firing-times. Fixed, limited ammunition supply.
Fundamental duel with kill probabilities dependent on
round number. General solution. For special case of
unlimited ammunition, general solution, and examples.
Introduces use of the differential-difference technique.

{15} N. K. Jaiswal and N. Bhashyam, "Stochastic Duels with Flight Time and

Replenishment," Defence Science Laboratory, Delhi, to be published.

Fundamental duel with random time-of-flight included.
No-delay procedure. Fixed initial ammunition supply.
Fixed ammunition replenishment at exponential times.
Random firing-times. General Laplace transform of time for
marksman to hit a passive target. General solution to
duel. Example. Introduces use of the Keilson-Kooharian
supplementary variable technique.

II. Multiple Duels

{16} See {5} above.

{17} See {6} above.

{18} See {2} above.

{19} Jane Ingersoll Robertson, "A Method of Computing Survival Probabilities

of Several Targets Versus Several Weapons," Operations Research, Vol. 4,

No. 5, October 1956, pp. 546-557.

> Several A's fire simultaneously and randomly at several
> B's, all with same kill probabilities. Survival
> probability computation methods.

{20} Robert L. Helmbold, "A 'Universal' Attrition Model," Operations Research,

Vol. 14, No. 4, July-August 1966, pp. 624-635.

> Several A's fire simultaneously and randomly at several
> B's, all with same kill probabilities which may vary
> from round to round. Intervisibility effects. Survival
> probabilities. Discrete approximations. Continuous
> analogues. Connection with Lanchester's mod·ˈ

DOCUMENT CONTROL DATA - R&D		
(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)		

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
System Development Corporation Santa Monica, California	Unclassified
	2b. GROUP

3. REPORT TITLE

The Status of Developments in the Theory of Stochastic Duels - II

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (Last name, first name, initial)

Ancker, C. J.

6. REPORT DATE 28 Sept 66	7a. TOTAL NO. OF PAGES 32	7b. NO. OF REFS 25

8a. CONTRACT OR GRANT NO. Independent Research	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	SP-1017/008/01
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited

11. SUPPLEMENTARY NOTES Supersedes SP-1017/008/00 by C.J.Ancker, dated 1 September 1964; AD-450 774	12. SPONSORING MILITARY ACTIVITY

13. ABSTRACT

This paper contains a brief outline of the history and the principal features of the Theory of Stochastic Duels, as well as the major analytical results to date. Areas for future research and possible fields of application are also considered. The connection with other models of combat, such as Lanchester's equations and games of strategy, is discussed. An annotated bibliography is included.

DD FORM 1473
1 JAN 64

14.		LINK A		LINK B		LINK C	
KEY WORDS		ROLE	WT	ROLE	WT	ROLE	WT
Stochastic Duels Combat Models ·Lanchester Equations Game Theory							

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization (*corporate author*) issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers (*either by the originator or by the sponsor*), also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

   (1) "Qualified requesters may obtain copies of this report from DDC."

   (2) "Foreign announcement and dissemination of this report by DDC is not authorized."

   (3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through

      _____ ."

   (4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through

      _____ ."

   (5) "All distribution of this report is controlled. Qualified DDC users shall request through

      _____ ."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring (*paying for*) the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as (TS). (S). (C), or (U).

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.